

# Pseudo Differential Operators and Markov Semigroups on Compact Lie Groups

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## Abstract

We extend the Ruzhansky-Turunen theory of pseudo differential operators on compact Lie groups into a tool that can be used to investigate group-valued Markov processes in the spirit of the work in Euclidean spaces of N.Jacob and collaborators. Feller semigroups, their generators and resolvents are exhibited as pseudo-differential operators and the symbols of the operators forming the semigroup are expressed in terms of the Fourier transform of the transition kernel. The symbols are explicitly computed for some examples including the Feller processes associated to stochastic flows arising from solutions of stochastic differential equations on the group driven by Lévy processes. We study a family of Lévy-type linear operators on general Lie groups that are pseudo differential operators when the group is compact and find conditions for them to give rise to symmetric Dirichlet forms.

*Key words and phrases.* Feller semigroup, pseudo differential operator, symbol, Fourier transform, Peter Weyl theorem, Lie group, Lie algebra, convolution semigroup, Courrège-Hunt operator, Sobolev space, Dirichlet form, Beurling-Deny representation.

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# 1 Introduction

The theory of pseudo differential operators has been a major theme within modern analysis since the early 1960s and there are important applications e.g. to partial differential equations, singular integral operators and index theory (see e.g. [43] and [22] for details.) In the last twenty years the theory has begun to interact with the study of stochastic processes, principally through the work of Niels Jacob and his collaborators. A key starting point is that the Feller semigroup and infinitesimal generator of a convolution semigroup of probability measures on  $\mathbb{R}^n$  are naturally represented as pseudo differential operators and the symbols of these operators may be obtained from the classical Lévy-Khintchine formula. This case is somewhat special in that the symbol is a function of the Fourier variable only and this corresponds to the fact that the characteristics of the measures have no state space dependence. The key insight of Jacob was to consider pseudo differential operators having more complicated symbols wherein the structure given by the Lévy-Khintchine formula is retained but the characteristics are now functions on  $\mathbb{R}^n$ . The programme is then to construct Feller-Markov processes whose associated transition semigroups have generators coinciding with these pseudo differential operators on a suitable domain. This was carried out by using the Hille-Yosida-Ray theorem in [25] and by solving the associated martingale problem in [19, 20]. Applications to the construction of Dirichlet forms may be found in [21]. A systematic development of this theory can be found in the three-volume work [27, 28, 29] (see also the earlier concise and very readable account in [26].)

The concept of pseudo differential operator may be extended to manifolds by using local co-ordinates, however when the manifold is a Lie group there is an alternative global formulation due to M. Ruzhansky and V. Turunen. Recall that if  $f \in L^1(\mathbb{R}^n, \mathbb{C})$  and its Fourier transform  $\widehat{f}$  is also integrable then we have the Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iu \cdot x} \widehat{f}(u) du,$$

and if  $\mathcal{A}$  is a pseudo differential operator with symbol  $\sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$  then

$$\mathcal{A}f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iu \cdot x} \sigma(x, u) \widehat{f}(u) du. \quad (1.1)$$

Now observe that  $\mathbb{R}^n$  is its own dual group and that for each  $u \in \mathbb{R}^n$ , the character  $e^{i(u, \cdot)}$  is an irreducible representation of  $\mathbb{R}^n$ . We can extend these concepts to arbitrary compact groups (Lie structure is not strictly needed here) by replacing the characters with general irreducible representations,

defining the symbol  $\sigma$  of a pseudo differential operator to be a suitable mapping defined on  $G \times \widehat{G}$  (where  $\widehat{G}$  is the set of all irreducible representations of  $G$  modulo unitary equivalence) and utilising the Fourier inversion formula of Peter-Weyl theory (see section 3 below for details.) Note that for each  $g \in G, \pi \in \widehat{G}, \sigma(g, \pi)$  will be a matrix acting on the same finite-dimensional complex vector space as  $\pi(g)$ . A detailed development of this theory can be found in the monograph [36], see also [37] for recent developments including applications to global hypoellipticity.

The purpose of the current paper is to begin the programme of developing the Ruzhansky-Turunen theory of pseudo differential operators on compact groups into a tool to study Markov processes in the spirit of Jacob's work on Euclidean space. In fact this task was started in the recent paper [5] but we only worked therein with convolution semigroups of measures (i.e. group-valued Lévy processes.) In the case where the group is abelian, there was also some work in this direction during the 1980s by N.Jacob [24]. Our first observation is that when we move to more general Markov processes we need to slightly extend the definition of pseudo differential operator from [36] where it is assumed that the operator preserves smooth functions. Essentially we will work in an  $L^2$  rather than a  $C^\infty$  framework. We motivate and present the relevant definitions in section 3 of the paper and also take the opportunity to review key ideas and concepts from [36].

The organisation of the paper is as follows. In section 2 we collect together some useful results about Feller semigroups in general locally compact spaces. As discussed above section 3 contains the definition and basic properties we'll need of general pseudo differential operators on compact groups. In section 3 we show that every Feller semigroup consists of pseudo differential operators and that these can be expressed in terms of the Fourier transform of the transition probability kernel of the associated process. In [5] (see also [3]) it was shown that the symbols of a convolution semigroup themselves form a contraction semigroup of matrices. Furthermore the symbol of the generator of the Feller semigroup is precisely the generator of the semigroup formed by the symbols. We show here that the convolution structure is also a necessary condition for the symbols to enjoy the semigroup property. Nonetheless in the more general case, there is still sufficient regularity for us to be able to capture the symbol of the generator by differentiating that of the semigroup at the origin.

In section 4, we collect together a number of examples of Feller processes on compact groups where we can calculate the symbol explicitly. These include Feller's pseudo-Poisson process, semigroups associated to stochastic flows obtained by solving stochastic differential equations on the group that

are driven by Lévy processes and processes that are obtained by subordination. Finally in section 5, we study a general class of Lévy-type linear operators to which we can associate a symbol and thus consider as pseudo differential operators. We call these *Courrège-Hunt operators* as they are obtained by replacing the characteristics in Hunt's celebrated formula [23] for the infinitesimal generator of a convolution semigroup with functions on the group in the spirit of Courrège's general form of linear operators which satisfy the positive maximal principle ([11, 12]). Note however that the local part of the operator in [12] is expressed in local co-ordinates while we employ the Lie algebra of the group to work globally (see [4] for an earlier less general approach to this problem.) We may also regard Courrège-Hunt operators as natural generalisations to Lie groups of the Lévy-type operators first studied by Komatsu [31] and Stroock [42] in the 1970s.

We analyse these operators in general Lie groups (no compactness assumption is necessary at this stage) and show that (under technical conditions) they are bounded operators from the second order Sobolev space to  $L^2(G)$  (the  $L^2$  space of Haar measure.). Under further technical conditions, we are able to explicitly compute the adjoint and find sufficient conditions for symmetry that naturally generalise the case where the operator is the generator of a convolution semigroup (see [5], [32].) Finally we are able to associate a symmetric Dirichlet form to the operator and hence realise it as the (restriction of) the generator of a Hunt process. We emphasise that the conditions for symmetry in  $L^2(G)$  are rather strong. For future work on symmetric Courrège-Hunt operators it may be worthwhile to seek infinitesimal invariant measures (see [1]) and work in the corresponding  $L^2$  space.

*Notation.* Let  $E$  be a locally compact topological space. The Borel  $\sigma$ -algebra of  $E$  will be denoted by  $\mathcal{B}(E)$ . We write  $B_b(E)$  for the real linear space of all bounded Borel measurable real-valued functions on  $E$ . It becomes a Banach space under the supremum norm  $\|f\|_\infty := \sup_{x \in E} |f(x)|$ . The closed subspace of  $B_b(E)$  comprising continuous functions will be denoted  $C_b(E)$  and  $C_0(E)$  will designate the closed subspace of continuous functions which vanish at infinity. The identity operator on  $C_0(E)$  is denoted by  $I$ . In the case where  $E$  is a finite dimensional real manifold,  $C^k(E)$  will denote the linear space of  $k$ -times continuously differentiable functions on  $E$ . The smooth functions of compact support on  $E$  are denoted by  $C_c^\infty(E)$  and these are a dense linear manifold in  $C_0(E)$ . We will need both real and complex functions spaces in this paper. If no field is explicitly indicated you should assume that we are working in a real space. A real valued function from  $[0, \infty)$  to  $E$  is said to be càdlàg if it is right continuous and left limits exist at each point.

If  $A$  is a set in a topological space then  $\overline{A}$  denotes its closure. Similarly  $\overline{T}$  will denote the closure of a closable linear operator  $T$  defined on a linear submanifold of a Banach space. If  $V$  is a complex Banach space then  $\mathcal{L}(V)$  will denote the algebra of all continuous linear operators on  $V$ . We will employ the Einstein summation convention where appropriate.

## 2 Feller Processes and Semigroups

Let  $(\Omega, \mathcal{F}, P)$  be a probability space wherein the  $\sigma$ -algebra  $\mathcal{F}$  is equipped with a filtration  $(\mathcal{F}_t, t \geq 0)$  of sub- $\sigma$ -algebras. Let  $Z = (Z(t), t \geq 0)$  be a time-homogeneous Markov process with respect to the given filtration with state space a locally compact topological space  $E$ . We will denote the random variable  $Z(t)$  by  $Z_x(t)$  when we are given the initial datum  $Z_0 = x$  (a.s.) where  $x \in E$ . We recall that the transition probabilities of the process are defined by  $p_t(x, A) := P(Z(t) \in A | Z(0) = x)$  for each  $t \geq 0, x \in E, A \in \mathcal{B}(E)$  and we assume that these are such that the mappings  $x \rightarrow p_t(x, A)$  are measurable for each  $t$  and  $A$ . We then get a one-parameter contraction semigroup  $(T_t, t \geq 0)$  on  $B_b(E)$  via the prescription

$$\begin{aligned} T_t f(x) &= \mathbb{E}(f(Z(t)) | Z(0) = x) \\ &= \int_E f(y) p_t(x, dy). \end{aligned}$$

This semigroup is also conservative in the sense that  $T_t 1 = 1$  for each  $t \geq 0$ . We say that  $(T_t, t \geq 0)$  is a *Feller semigroup* (in which case  $Z$  is said to be a Feller process) if

$$(F1) \quad T_t(C_0(E)) \subseteq C_0(E) \text{ for each } t \geq 0,$$

$$(F2) \quad \lim_{t \rightarrow 0} \|T_t f - f\|_\infty = 0.$$

In this case  $(T_t, t \geq 0)$  is a strongly continuous contraction semigroup on  $C_0(E)$  and we denote its infinitesimal generator by  $\mathcal{A}$ . We will also require the resolvent  $R_\lambda(\mathcal{A}) := (\lambda I - \mathcal{A})^{-1}$  which is a bounded operator on  $C_0(E)$  for all  $\lambda > 0$ .

Note that to establish (F2) it is sufficient (by a standard  $\frac{\epsilon}{3}$ -argument) to verify that the limit vanishes for arbitrary  $f$  in a dense linear subspace of  $C_0(E)$  and we will use this fact in the sequel without further comment.

Let  $Z$  be a Markov process. We say that there is an associated *solution map* if for each  $t \geq 0$  there exists a measurable mapping  $\Phi_t : E \times \Omega \rightarrow \Omega$  such that  $\Phi_t(x, \omega) = Z_x(t)(\omega)$  for all  $x \in E, \omega \in \Omega$ . This structure typically arises

when  $Z$  is the solution of a stochastic differential equation (SDE) wherein the coefficients are sufficiently regular for the solution to generate a stochastic flow of homeomorphisms. The following result is surely well-known but we include a brief proof to make the paper more complete. We will need it in section 5 of the paper when we consider examples in compact groups.

**Theorem 2.1** *Suppose that  $E$  is compact. Let  $Z = (Z(t), t \geq 0)$  be a homogeneous Markov process for which there is an associated solution map which is such that*

1. *The mapping  $\Phi_t : E \rightarrow E$  is continuous (a.s) for all  $t \geq 0$ .*
2.  *$Z$  solves the “martingale problem” in that there exists a densely defined linear operator  $A$  in  $C_0(E)$  with domain  $D_A$  such that for all  $f \in D_A, x \in E, t \geq 0$*

$$f(\Phi_t(x)) - f(x) - \int_0^t Af(\Phi_s(x))ds \text{ is a centred martingale.} \quad (2.2)$$

*Then  $(Z(t), t \geq 0)$  is a Feller process and  $A$  extends to the infinitesimal generator of the associated Feller semigroup.*

*Proof.* Let  $(T_t, t \geq 0)$  be the semigroup associated to the process  $Z$  acting on  $B_b(E)$ . Let  $(x_n, n \in \mathbb{N})$  be a sequence in  $E$  which converges to  $x$ . Then for any  $f \in C_0(E)$ ,

$$\lim_{n \rightarrow 0} |T_t f(x) - T_t f(x_n)| = \lim_{n \rightarrow 0} |\mathbb{E}(f(\Phi_t(x))) - f(\Phi_t(x_n))| = 0,$$

by a straightforward application of dominated convergence and so (F1) is satisfied. For (F2) it is sufficient to take expectations in (2.2) to obtain, for all  $f \in D_A, x \in E, t \geq 0$ :

$$T_t f(x) - f(x) = \int_0^t T_s A f(x) ds,$$

and the result follows easily.  $\square$

From now on we will use the term *Feller semigroup* to mean any strongly continuous contraction semigroup of positive operators on  $C_0(E)$ .

Now let  $B$  be a linear operator on  $C_0(E)$  with domain  $D_B$ . We say that  $B$  satisfies the *positive maximum principle* if  $f \in D_B$  and  $f(x_0) = \sup_{x \in E} f(x) \geq 0$  implies  $Bf(x_0) \leq 0$ . It is well-known and easily verified that if  $A$  is the infinitesimal generator of a Feller semigroup then it satisfies the positive maximum principle. Conversely we have

**Theorem 2.2 (Hille-Yosida-Ray)** *A linear operator  $A$  defined on  $C_0(E)$  with domain  $D_A$  is closable and its closure is the infinitesimal generator of a Feller semigroup if and only if*

- (a)  $D_A$  is dense in  $C_0(E)$ .
- (b)  $A$  satisfies the positive maximum principle.
- (c)  $\text{Ran}(\lambda I - A)$  is dense in  $C_0(E)$  for some  $\lambda > 0$ .

See e.g. [13] pp.165-6 for a proof of this result.

Explicit characterisations of linear operators that satisfy the positive maximum principle can be found in [11] in the case  $E = \mathbb{R}^n$ , in [12] when  $E$  is a compact manifold and in [10] when  $E$  is a manifold with compact boundary.

A Feller semigroup  $(T_t, t \geq 0)$  is said to be *conservative* if its action on  $B_b(E)$  is such that  $T_t 1 = 1$  for all  $t \geq 0$ . A sufficient condition for this is that  $(1, 0)$  lies in the bp-closure of the graph of  $A$  (see e.g. Lemma 2.1 in [39]). If  $E$  is compact (as is the case for most of this paper) then a necessary and sufficient condition is that  $1 \in \text{Dom}(A)$  and  $A1 = 0$ .

If  $E$  is also separable then given any conservative Feller semigroup we can construct an associated Markov process with given initial distribution and having càdlàg paths (see Theorem 2.7 in [13], pp 169-70).

### 3 Pseudo Differential Operators on Compact Groups

Let  $G$  be a compact group with neutral element  $e$  and let  $\widehat{G}$  be the set of equivalence classes (with respect to unitary isomorphism) of irreducible representations of  $G$ . We will often identify equivalence classes  $[\pi]$  with a typical element  $\pi$ . So for each  $g \in G$ ,  $\pi(g)$  is a unitary matrix acting on a finite dimensional complex vector space  $V_\pi$  having dimension  $d_\pi$ . We fix an orthonormal basis  $(e_i^{(\pi)}, 1 \leq i \leq d_\pi)$  in  $V_\pi$  and define the co-ordinate functions  $\pi_{ij}(g) = \langle \pi(g)e_i^{(\pi)}, e_j^{(\pi)} \rangle$  for each  $1 \leq i, j \leq d_\pi, g \in G$ . Let  $L^2(G, \mathbb{C}) := L^2(G, m; \mathbb{C})$  where  $m$  is normalised Haar measure on  $(G, \mathcal{B}(G))$ . The celebrated Peter-Weyl theorem tells us that  $\{d_\pi^{-\frac{1}{2}}\pi_{ij}, 1 \leq i, j \leq d_\pi\}$  is a complete orthonormal basis for  $L^2(G, \mathbb{C})$ . Moreover  $L^2(G, \mathbb{C}) = \overline{\mathcal{M}}$  where  $\mathcal{M}$  is the linear span of  $\{h_\pi(\psi, \phi), \psi, \phi \in V_\pi, \pi \in \widehat{G}\}$  where for each  $g \in G$ ,  $h_\pi(\psi, \phi)(g) := \langle \pi(g)\psi, \phi \rangle$ . Furthermore  $\mathcal{M}$  is also dense in  $C(G)$  (with the usual topology of uniform convergence.) See e.g. [14] for details. In the sequel we will find it convenient to sometimes extend the notation  $h_\pi$

to situations where  $\psi$  is replaced by a function from  $G$  to  $V_\pi$  which is such that the mappings  $g \rightarrow \langle \psi(g), \phi \rangle$  are measurable for each  $\phi \in V_\pi$ . In this case we define  $h_\pi(\psi(\cdot), \phi)$  to be the measurable function from  $G$  to  $\mathbb{C}$  for which  $h_\pi(\psi(\cdot), \phi)(g) = \langle \pi(g)\psi(g), \phi \rangle$  for each  $g \in G$ .

The Fourier transform of  $f \in L^2(G, \mathbb{C})$  is defined by

$$\widehat{f}(\pi) = \int_G \pi(g^{-1})f(g)dg,$$

where we use  $dg$  as shorthand for  $m(dg)$  and Fourier inversion then yields

$$f = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi)\pi), \quad (3.3)$$

in the  $L^2$ -sense. In particular for almost all  $g \in G$

$$f(g) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi)\pi(g)).$$

Fourier transforms of distributions on  $G$  are defined by using duality (see e.g. [36] p.545)

Now let  $\mathcal{R}(\widehat{G}) = \bigcup_{\pi \in \widehat{G}} V_\pi$ . Let  $C^\infty(G, \mathbb{C})$  be equipped with its uniform topology and let  $A : C^\infty(G, \mathbb{C}) \rightarrow C^\infty(G, \mathbb{C})$  be a bounded linear map. The Ruzhansky-Turunen theory of pseudo differential operators is based on the following ideas (see [36], p.552). By the Schwartz kernel theorem we may write  $A$  as a convolution operator in that there exists a distribution on  $G \times G$  such that for each  $f \in C^\infty(G, \mathbb{C})$ ,  $g \in G$

$$(Af)(g) = \int_G f(h)R(g, h^{-1}g)dh.$$

We define the *symbol* of  $A$  to be the mapping  $\sigma_A : G \times \widehat{G} \rightarrow \mathcal{R}(\widehat{G})$  defined by  $\sigma_A(g, \pi) = \widehat{r}_g(\pi)$  where  $r_g(\cdot) = R(g, \cdot)$  so that  $\sigma_A(g, \pi) \in \mathcal{L}(V_\pi)$  for each  $g \in G, \pi \in \widehat{G}$ .

We then have (see Theorem 10.4.4 in [36], pp.552-3)

$$Af(g) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\sigma_A(g, \pi)\widehat{f}(\pi)\pi(g)), \quad (3.4)$$

for all  $f \in C^\infty(G, \mathbb{C})$ ,  $g \in G$ .

It then follows (see [36], Theorem 10.4.6, pp 553-4) that for all  $1 \leq i, j \leq d_\pi$ ,  $\pi \in \widehat{G}$ ,  $g \in G$ :



$$\sigma_A(g, \pi)_{ij} = \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(g)} A \pi_{kj}(g), \quad (3.5)$$

which we can write succinctly as

$$\sigma_A(g, \pi) = \pi(g^{-1}) A \pi(g), \quad (3.6)$$

with the understanding that (3.6) is a convenient shorthand for the precise statement (3.5). These ideas extend naturally to the more general case where  $A$  maps  $C^\infty(G, \mathbb{C})$  to the space of all distributions on  $G$  (see [37].)

We want to extend the pseudo differential operator framework to more general operators that are encountered in Markov process theory. In [5] where we essentially only dealt with Lévy processes, the symbols of all the operators that we considered were “functions” of  $\pi$  only and this allowed us to effectively use (3.4) as our definition but within a  $L^2$  framework. Now we want to discuss Feller processes, it seems that a more convenient definition (and certainly one that is in line with the theory of [26] and [27]) is to take (3.6) (or equivalently (3.5)) as our starting point. To be precise we say that a linear operator  $A$  defined on  $L^2(G, \mathbb{C})$  is a *pseudo differential operator* if

- (PD1)  $\mathcal{M} \subseteq \text{Dom}(A)$
- (PD2) There exists a mapping  $\sigma_A : G \times \widehat{G} \rightarrow \mathcal{R}(\widehat{G})$  such that  $\sigma_A(g, \pi) \in \mathcal{L}(V_\pi)$  for each  $g \in G, \pi \in \widehat{G}$  with  $\sigma_A(g, \pi) = \pi(g^{-1}) A \pi(g)$  for all  $\pi \in \widehat{G}, g \in G$ .
- (PD3) For each  $\pi \in \widehat{G}$ , the mapping  $g \rightarrow \sigma_A(g, \pi)$  is weakly (or equivalently, strongly) measurable.

Of course we still call  $\sigma_A$  the symbol of  $A$ . Note that a necessary and sufficient condition for (PD3) is that each matrix entry of  $\sigma_A(g, \pi)$  is a measurable complex-valued function of  $g \in G$ .

Note that its symbol determines the action of a pseudo differential operator uniquely on  $\mathcal{M}$ . To see this suppose that  $A_1$  and  $A_2$  are densely defined linear operators in  $L^2(G)$  with  $\mathcal{M} \subseteq \text{Dom}(A_1) \cap \text{Dom}(A_2)$  and that  $\sigma_{A_1}(g, \pi) = \sigma_{A_2}(g, \pi)$  for all  $g \in G, \pi \in \widehat{G}$ . Then it follows from (3.5) that  $A_1 \pi_{ij} = A_2 \pi_{ij}$  for all  $1 \leq i, j \leq d_\pi, \pi \in \widehat{G}$  and hence by linearity,  $A_1 \psi = A_2 \psi$  for all  $\psi \in \mathcal{M}$ .

We can recover the Fourier inversion representation (3.4) in our theory as follows.

**Proposition 3.1** *Let  $A$  be a pseudo differential operator on  $L^2(G, \mathbb{C})$  with symbol  $\sigma_A$ . Suppose that  $A$  is closed and that  $\sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi) A \pi)$  converges in  $L^2(G, \mathbb{C})$  for all  $f \in \text{Dom}(A)$ . Then*

$$Af(g) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\sigma_A(g, \pi) \widehat{f}(\pi) \pi(g)), \quad (3.7)$$

for almost all  $g \in G$ .

*Proof.* By Fourier inversion (3.3) we have for  $f \in \text{Dom}(A)$

$$f = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ij} \pi_{ji}.$$

It follows from the fact that  $A$  is closed that for almost all  $g \in G$ ,

$$\begin{aligned} Af(g) &= \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ij} A \pi_{ji}(g) \\ &= \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi) A \pi(g)) \\ &= \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi) \pi(g) \sigma_A(g, \pi)) \\ &= \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\sigma_A(g, \pi) \widehat{f}(\pi) \pi(g)), \end{aligned}$$

where we have used (3.6) and the fact that  $\text{tr}(XY) = \text{tr}(YX)$ .  $\square$

In the sequel, when we consider Feller semigroups, we will want to consider operators that preserve the space of continuous functions on  $G$ . Indeed we will need the fact that any densely defined linear operator on  $C(G)$  (equipped with the uniform topology) is also densely defined on  $L^2(G)$ .

**Proposition 3.2** *If  $A : C(G) \rightarrow C(G)$  is a pseudo differential operator then for each  $\pi \in G$  the mapping  $g \rightarrow \sigma_A(g, \pi)$  is weakly (equivalently, strongly) continuous.*

*Proof.* This follows immediately from (3.5).  $\square$

The following result gives an equivalent characterisation of a pseudo differential operator:

**Proposition 3.3** *A linear operator  $A$  defined on  $L^2(G, \mathbb{C})$  with  $\mathcal{M} \subseteq \text{Dom}(A)$  is a pseudo differential operator if and only if there exists a mapping  $\sigma_A : G \times \widehat{G} \rightarrow \mathcal{R}(\widehat{G})$  satisfying (PD3) such that  $\sigma_A(g, \pi) \in \mathcal{L}(V_\pi)$  and*

$$Ah_\pi(\psi, \phi)(g) = h_\pi(\sigma_A(\cdot, \pi)\psi, \phi)(g), \quad (3.8)$$

for all  $h_\pi \in \mathcal{M}, g \in G$ .

*Proof.* With  $h_\pi, g$  as in the statement of the proposition, by (3.6):

$$\begin{aligned} Ah_\pi(\psi, \phi)(g) &= \langle A\pi(g)\psi, \phi \rangle \\ &= \langle \pi(g^{-1})A\pi(g)\psi, \pi(g^{-1})\phi \rangle \\ &= \langle \sigma_A(g, \pi)\psi, \pi(g^{-1})\phi \rangle \\ &= \langle \pi(g)\sigma_A(g, \pi)\psi, \phi \rangle \\ &= h_\pi(\sigma_A(\cdot, \pi)\psi, \phi)(g), \end{aligned}$$

where we have used the fact that  $\pi(g)^* = \pi(g^{-1})$ . The converse follows when we take  $\psi$  and  $\phi$  to be  $e_i^{(\pi)}$  and  $e_j^{(\pi)}$  (respectively). We then find by (3.8) that

$$A\langle \pi(\cdot)e_i^{(\pi)}, e_j^{(\pi)} \rangle(g) = \langle \pi(g)\sigma_A(g, \pi)e_i^{(\pi)}, e_j^{(\pi)} \rangle$$

and so

$$A\pi_{ij}(g) = \sum_{k=1}^{d_\pi} \pi_{ik}(g)\sigma_A(g, \pi)_{kj}.$$

By unitarity of the matrix  $\pi(g)$  for each  $1 \leq l \leq d_\pi$ , we obtain

$$\sum_{i=1}^{d_\pi} \overline{\pi_{il}(g)} A\pi_{ij}(g) = \sigma_A(g, \pi)_{lj},$$

and since this holds for each  $1 \leq j \leq d_\pi$  then (3.5) is obtained, as required.  $\square$

We remark that Proposition 3.3 enables us to extend the definition of pseudo differential operators to more general topological groups. We will not pursue that theme further in this article.

## 4 Pseudo Differential operators and Feller semi-groups

We begin this section with some preliminaries about measures on groups and their Fourier transforms. Recall (see e.g. [16], [41]) that if  $\mu$  is a Borel

probability measure defined on  $G$  then its Fourier transform  $(\widehat{\mu}(\pi), \pi \in \widehat{G})$  is defined by the matrix-valued integral<sup>1</sup>:

$$\widehat{\mu}(\pi) = \int_G \pi(g) \mu(dg),$$

so that each  $\widehat{\mu}(\pi)$  acts as a contraction on  $V_\pi$ . Furthermore  $\widehat{\mu}$  determines  $\mu$  uniquely. If  $\mu$  and  $\nu$  are Borel probability measures on  $G$  then for all  $\pi \in \widehat{G}$

$$\widehat{\mu * \nu}(\pi) = \widehat{\mu}(\pi) \widehat{\nu}(\pi),$$

where  $\mu * \nu$  denotes the convolution of  $\mu$  and  $\nu$ , i.e. the unique Borel probability measure on  $G$  for which

$$\int_G f(\rho) (\mu * \nu)(d\rho) = \int_G \int_G f(\rho\tau) \mu(d\rho) \nu(d\tau),$$

for all  $f \in B_b(G)$ .

Let  $(T_t, t \geq 0)$  be a Feller semigroup defined on  $C(G)$  where  $G$  is a compact group. Then for each  $t \geq 0$ ,  $T_t$  is a densely defined (not necessarily bounded) linear operator in  $L^2(G)$  with  $\mathcal{M} \subseteq \text{Dom}(T_t)$ . Recall that for each  $t \geq 0, g \in G$  the transition probability  $p_t(g, \cdot)$  is a Borel probability measure on  $G$ . We denote its Fourier transform at  $\pi \in \widehat{G}$  by  $\widehat{p}_t(g, \pi)$ .

**Lemma 4.1** *For each  $t \geq 0$ ,  $T_t$  is a pseudo differential operator with symbol*

$$\sigma_t(g, \pi) = \pi(g^{-1}) \widehat{p}_t(g, \pi), \tag{4.9}$$

for  $g \in G, \pi \in \widehat{G}$ .

*Proof.* From the discussion above it follows that (PD1) is satisfied. To establish the identity (4.9) we use

$$\begin{aligned} \sigma_t(g, \pi) &= \pi(g^{-1}) T_t \pi(g) \\ &= \pi(g^{-1}) \int_G \pi(\tau) p_t(g, d\tau) \\ &= \pi(g^{-1}) \widehat{p}_t(g, \pi), \end{aligned}$$

as required. (PD3) follows from the facts that  $g \rightarrow \widehat{p}_t(g, \pi)$  is weakly measurable and  $g \rightarrow \pi(g)$  is strongly continuous.  $\square$

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<sup>1</sup>Note that we employ the “probabilist’s convention” for Fourier transforms of measures and the “analyst’s convention” for those of functions. Readers should be reassured that this does not result in any mathematical inconsistency.

Note that we always have  $\sigma_0(g, \pi) = I_\pi$ , where  $I_\pi$  is the identity matrix on  $V_\pi$ .

*Example. Convolution Semigroups.* Let  $(\mu_t, t \geq 0)$  to be a vaguely continuous convolution semigroup of probability measures on  $G$ . Such semigroups of measures arise as the laws of  $G$ -valued Lévy processes (see e.g. [33]). In this case for each  $t \geq 0, g \in G, f \in C(G), T_t f(g) = \int_G f(g\tau) \mu_t(d\tau)$  and  $p_t(g, A) = \mu_t(g^{-1}A)$  for each  $g \in G, A \in \mathcal{B}(G)$ . So by Lemma 4.1 we obtain

$$\begin{aligned}\sigma_t(g, \pi) &= \pi(g^{-1}) \int_G \pi(\tau) \mu_t(g^{-1}d\tau) \\ &= \widehat{\mu}_t(\pi)\end{aligned}$$

for all  $g \in G, \pi \in \widehat{G}$ . It is shown in [3] that for each  $\pi \in G, (\widehat{\mu}_t(\pi), t \geq 0)$  is a strongly continuous contraction semigroup of matrices acting on  $V_\pi$ . The fact that the symbol is independent of  $g \in G$  is clearly related to the translation invariance of the semigroup, i.e. the fact that  $L_\rho T_t = T_t L_\rho$  for all  $\rho \in G, t \geq 0$  where  $L_\rho f(g) = f(\rho^{-1}g)$  for each  $f \in C(G), g \in G$  (c.f. [23].)

We will investigate more examples in the next section. First we establish some general properties.

**Theorem 4.1** *Let  $(\sigma_t, t \geq 0)$  be the symbol of a Feller semigroup  $(T_t, t \geq 0)$  with infinitesimal generator  $\mathcal{A}$  and assume that  $\mathcal{M} \subseteq \text{Dom}(\mathcal{A})$ . Then for each  $g \in G, \pi \in \widehat{G}$ ,*

1. *The mapping  $t \rightarrow \sigma_t(g, \pi)$  is strongly differentiable in  $V_\pi$ .*
2. *For each  $s, t \geq 0$ ,*

$$\sigma_{s+t}(g, \pi) = \pi(g^{-1}) T_s \widehat{p}_t(\cdot, \pi)(g). \quad (4.10)$$

*Proof.*

1. Since  $(T_t, t \geq 0)$  is strongly differentiable on  $\text{Dom}(\mathcal{A})$  and  $\pi_{ij} \in \text{Dom}(\mathcal{A})$  for each  $1 \leq i, j \leq d_\pi$ , it follows by (3.5) that each matrix entry  $\sigma_t(g, \pi)_{ij}$  is a differentiable function of  $t$ . We define a  $d_\pi \times d_\pi$  matrix  $j_t(g, \pi)$  by the prescription

$$j_t(g, \pi)_{ij} = \frac{d}{dt} \sigma_t(g, \pi)_{ij} = \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(g)} T_t \mathcal{A} \pi_{kj}(g).$$

Now let  $\psi = (\psi_1, \dots, \psi_{d_\pi}) \in V_\pi$ . Using the Cauchy-Schwarz inequality (twice) and the fact that  $|\pi_{ij}(g)| \leq 1$  for each  $1 \leq i, j \leq d_\pi, g \in G$  we have for each  $t \geq 0$

$$\begin{aligned}
& \limsup_{h \rightarrow 0} \frac{1}{h} \|\sigma_{t+h}(g, \pi)\psi - \sigma_t(g, \pi)\psi - h j_t(g, \pi)\psi\|_{V_\pi}^2 \\
&= \limsup_{h \rightarrow 0} \frac{1}{h} \sum_{i=1}^{d_\pi} \left| \sum_{j=1}^{d_\pi} \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(g)} (T_{t+h}\pi_{kj}(g) - T_t\pi_{kj}(g) - h T_t \mathcal{A}\pi_{kj}(g)) \psi_j \right|^2 \\
&\leq d_\pi \|\psi\|_{V_\pi}^2 \limsup_{h \rightarrow 0} \frac{1}{h} \sum_{i,j,k=1}^{d_\pi} |T_{t+h}\pi_{kj}(g) - T_t\pi_{kj}(g) - h T_t \mathcal{A}\pi_{kj}(g)|^2 = 0,
\end{aligned}$$

and the result follows since  $\pi_{kj} \in \text{Dom}(\mathcal{A})$ .

2. Using the Chapman-Kolmogorov equations we find that

$$\begin{aligned}
\sigma_{s+t}(g, \pi) &= \pi(g^{-1}) T_{s+t} \pi(g) \\
&= \int_G \pi(g^{-1} \tau) p_{s+t}(g, d\tau) \\
&= \int_G \pi(g^{-1} \rho) \left( \int_G \pi(\rho^{-1} \tau) p_t(\rho, d\tau) \right) p_s(g, d\rho) \\
&= \int_G \pi(g^{-1} \rho) \sigma_t(\rho, \pi) p_s(g, d\rho) \\
&= \pi(g^{-1}) T_s (\pi(\cdot) \sigma_t(\cdot, \pi))(g) \\
&= \pi(g^{-1}) T_s \widehat{p}_t(\cdot, \pi)(g),
\end{aligned}$$

where we used (4.9) to obtain the last line.

□

For each  $g \in G, \pi \in \widehat{G}$  we define a  $d_\pi \times d_\pi$  matrix  $j(g, \pi)$  by the prescription

$$j(g, \pi)_{ij} := \frac{d}{dt} \sigma_t(g, \pi)_{ij} \Big|_{t=0} \quad (4.11)$$

for each  $1 \leq i, j \leq d_\pi$  (of course  $j$  is just  $j_0$  from the proof of Theorem 4.1.)

**Theorem 4.2** 1. If  $\mathcal{M} \subseteq \text{Dom}(\mathcal{A})$  then  $\mathcal{A}$  is a pseudo differential operator with symbol  $j(g, \pi)$  at  $\pi \in \widehat{G}, g \in G$ .

2. For each  $\lambda > 0, R_\lambda(A)$  is a pseudo differential operator with symbol  $\int_0^\infty e^{-\lambda t} \sigma_t(g, \pi) dt$  at  $\pi \in \widehat{G}, g \in G$ .

*Proof.*

1. Using (3.6) we compute

$$\begin{aligned}\pi(g^{-1})\mathcal{A}\pi(g) &= \pi(g^{-1}) \left. \frac{d}{dt} T_t \pi(g) \right|_{t=0} \\ &= \left. \frac{d}{dt} \pi(g^{-1}) T_t \pi(g) \right|_{t=0} \\ &= \left. \frac{d}{dt} \sigma_t(g, \pi) \right|_{t=0} = j(g, \pi).\end{aligned}$$

2. This follows from (3.6) using the fact that  $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$  for each  $f \in C(G)$ .  $\square$

**Note.** If  $(T_t, t \geq 0)$  is the semigroup associated to the  $G$ -valued Feller process  $(X(t), t \geq 0)$  then Theorem 4.2(1) tells us that for all  $g \in G, \pi \in \widehat{G}$ ,

$$j(g, \pi) = \lim_{t \rightarrow 0} \frac{1}{t} (\pi(g^{-1}) \mathbb{E}(\pi(X(t)) | X(0) = g) - I_\pi),$$

and this should be compared with the corresponding analysis in Euclidean space - see e.g. equation (0.3) on p.xx in [27] and Definition 3.9.1 on p.148 of [29].

We close this section by returning to the question of when the matrices  $(\sigma_t(e, \pi), t \geq 0)$  form a semigroup on  $V_\pi$ . We know from (4.10) that this is unlikely to hold in general, but that this is the case when the transition probabilities are translates of a vaguely continuous convolution semigroup of probability measures. The next result shows that this is the only possibility.

**Proposition 4.1** *For each  $t \geq 0$ , let  $\sigma_t(g, \pi)$  be the symbol of a Feller semigroup at  $g \in G, \pi \in \widehat{G}$ . We have  $\sigma_{s+t}(e, \pi) = \sigma_s(e, \pi) \sigma_t(e, \pi)$  if and only if  $(p_t(e, \cdot), t \geq 0)$  is a vaguely continuous convolution semigroup of probability measures on  $G$ .*

*Proof.* By (4.9)  $\sigma_t(e, \pi) = \widehat{p}_t(e, \pi)$  and so we have  $\sigma_{s+t}(e, \pi) = \sigma_s(e, \pi) \sigma_t(e, \pi)$  if and only if

$$\widehat{p_{s+t}}(e, \pi) = \widehat{p_s}(e, \pi) \widehat{p_t}(e, \pi) = \widehat{p_s * p_t}(e, \pi)$$

for all  $\pi \in \widehat{G}$ , and the result follows by uniqueness of the Fourier transform.  $\square$

## 5 Examples

### 1. Feller's Pseudo-Poisson Process

Let  $(S(n), n \in \mathbb{Z}_+)$  be a Markov chain taking values in  $G$  with transition kernel  $q : G \times \mathcal{B}(G) \rightarrow [0, 1]$ , so that  $q(g, B) = P(S(1) \in B | S(0) = g)$  for all  $g \in G, B \in \mathcal{B}(G)$ . Let  $(N(t), t \geq 0)$  be a Poisson process of intensity  $\lambda > 0$  that is independent of  $(S(n), n \in \mathbb{Z}_+)$ . Then Feller's pseudo-Poisson process  $(Z(t), t \geq 0)$  is defined by  $Z(t) = S(N(t))$  for each  $t \geq 0$ . It is well known (see e.g. [6], [13], [29]) that it is a Feller process with bounded infinitesimal generator defined for each  $f \in L^2(G, \mathbb{C}), g \in G$  by

$$\mathcal{A}f(g) = \lambda \int_G (f(\tau) - f(g))q(g, d\tau). \quad (5.12)$$

Hence by a straightforward application of the definition (PD2), we find that the symbol is given by

$$j(g, \pi) = \lambda \int_G (\pi(g^{-1}\tau) - I_\pi)q(g, d\tau) \quad (5.13)$$

for all  $\pi \in \widehat{G}$ .

For the remaining examples in this section, we will require that  $G$  be a compact  $n$ -dimensional Lie group with Lie algebra  $\mathfrak{g}$ . Let  $\exp$  be the exponential map from  $\mathfrak{g}$  to  $G$ . Then for each  $X \in \mathfrak{g}, \pi \in \widehat{G}$ , the skew-symmetric matrix  $d\pi(X)$  is defined by

$$d\pi(X) = \left. \frac{d}{du} \pi(\exp(uX)) \right|_{u=0}.$$

Now consider the left invariant vector field  $g \rightarrow X(g)$  for which  $X(e) = X$ . It is a pseudo differential operator and its symbol is given by

$$\begin{aligned} \sigma_X(g, \pi) &= \pi(g^{-1})X\pi(g) \\ &= \pi(g^{-1}) \left. \frac{d}{du} \pi(g \exp(uX)) \right|_{u=0} \\ &= \left. \frac{d}{du} \pi(\exp(uX)) \right|_{u=0} \\ &= d\pi(X), \end{aligned}$$

for each  $g \in G, \pi \in \widehat{G}$ . If  $X'$  is the right invariant vector field for which  $X'(e) = X$  then a similar argument yields  $\sigma_{X'}(g, \pi) = \pi(g^{-1})d\pi(X)\pi(g)$ . We will make use of these facts in the remaining examples.



## 2. Convolution Semigroups

Let  $(\mu_t, t \geq 0)$  be a vaguely continuous convolution semigroup of measures on  $G$  with  $\mu_0 = \delta_e$ . As discussed above the associated Feller semigroup on  $C(G)$  is given by  $T_t f(g) = \int_G f(g\tau) \mu_t(d\tau)$  for  $f \in C(G), t \geq 0, g \in G$ . The infinitesimal generator  $\mathcal{A}$  was first studied by Hunt [23] - see also [18] and [33]. Let  $\{X_1, \dots, X_n\}$  a fixed basis of  $\mathfrak{g}$  and define

$$C_2(G) := \{f \in C(G), X_i f \in C(G), X_j X_k f \in C(G), 1 \leq i, j, k \leq n\}.$$

It is easily verified that  $\mathcal{M} \subseteq C_2(G)$ . It is shown in [23] that  $C_2(G) \subseteq \text{Dom}(\mathcal{A})$  and for  $f \in C_2(G)$  and

$$\begin{aligned} \mathcal{A}f(g) &= b^i X_i f(g) + a^{ij} X_i X_j f(g) \\ &+ \int_{G-\{e\}} [f(g\tau) - f(g) - x^i(\tau) X_i f(g)] \nu(d\tau) \end{aligned} \quad (5.14)$$

where  $b \in \mathbb{R}^n, (a_{ij})$  is a non-negative definite symmetric  $n \times n$  matrix and  $\nu$  is a Lévy measure on  $G - \{e\}$  i.e. a Borel measure for which

$\int_{G-\{e\}} (\sum_{i=1}^n x_i(\tau)^2 \wedge 1) \nu(d\tau) < \infty$ . Here  $x_i \in C^\infty(G)$  for  $1 \leq i \leq n$  is such that  $(x_1, \dots, x_n)$  are canonical co-ordinates in a neighborhood of  $e$  with  $x_i(e) = 0$  and  $X_i x_j(e) = \delta_{ij} (1 \leq i, j \leq n)$ .

We have already seen that the symbol of  $T_t$  is  $\hat{\mu}_t$  for each  $t \geq 0$ . It follows from (4.11) that that of  $\mathcal{A}$  is the infinitesimal generator of the matrix semigroup  $(\hat{\mu}_t, t \geq 0)$ . This is obtained in [3] (see also [17], [34] and the discussion in [5]) and we have for all  $g \in G, \pi \in \hat{G}$

$$\begin{aligned} j(g, \pi) &= b^i d\pi(X_i) + a^{ij} d\pi(X_i) d\pi(X_j) \\ &+ \int_{G-\{e\}} [\pi(\tau) - I_\pi - x^i(\tau) d\pi(X_i)] \nu(d\tau) \end{aligned} \quad (5.15)$$

## 3. Lévy Flows and Diffusions

Let  $Y_1, \dots, Y_p$  be  $C^\infty$ -vector fields on  $G$  and let  $X_1, \dots, X_n$  be a basis for  $\mathfrak{g}$ . Since  $G$  is parallelisable, we can assert the existence of  $C^\infty$ -mappings  $\gamma_i^j : G \rightarrow \mathbb{R}$  such that for each  $g \in G, Y_i(g) = \gamma_i^j(g) X_j(g)$  for each  $1 \leq i \leq p$ . Now let  $(L(t), t \geq 0)$  be an  $\mathbb{R}^p$ -valued càdlàg Lévy process with characteristics  $(b, a, \nu)$  and consider the stochastic differential equation

$$d\phi(t) = Y_i(\phi(t-)) \diamond dL^i(t), \quad (5.16)$$

where  $\diamond$  denotes the Markus canonical form (see e.g. section 6.10 in [6].) It is shown in [8] pp. 233-4 that the solution map is a stochastic flow of diffeomorphisms of  $G$  and it follows by Theorem 2.1 that  $(\phi(t), t \geq 0)$  is a Feller process. The infinitesimal generator is given by

$$\begin{aligned} \mathcal{A}f(g) &= b^i Y_i f(g) + \frac{1}{2} a^{ij} Y_i Y_j f(g) \\ &+ \int_{\mathbb{R}^p - \{0\}} (f(\xi(y)(g)) - f(g) - y^i Y_i f(g) \mathbf{1}_{|y| < 1}) \nu(dy) \end{aligned} \quad (5.17)$$

for  $f \in C^2(G)$ ,  $g \in G$ . Here  $(\xi(y), y \in G)$  is the flow of diffeomorphisms of  $G$  obtained by solving the ordinary differential equation

$$\frac{d\xi(vy)}{dv} = y^i Y_i \xi(vy).$$

In this case the symbol of  $\mathcal{A}$  is given by

$$\begin{aligned} j(g, \pi) &= b^i \gamma_i^j(g) d\pi(X_j) \\ &+ \frac{1}{2} a^{ij} \gamma_i^k(g) [\gamma_j^r(g) d\pi(X_k) d\pi(X_r) + X_k(g) \gamma_j^r(g) d\pi(X_r)] \\ &+ \int_{\mathbb{R}^p - \{0\}} (\pi(g^{-1} \xi(y)(g)) - I_\pi - y^i \gamma_i^k(g) d\pi(X_k) \mathbf{1}_{|y| < 1}) \nu(dy), \end{aligned} \quad (5.18)$$

for each  $\pi \in \widehat{G}$ ,  $g \in G$ .

Note that when  $\nu \equiv 0$ , then  $\diamond$  is a Stratonovitch differential equation and  $(\phi(t), t \geq 0)$  gives rise to a diffusion process on  $G$  (see [9] for some interesting recent work on such processes.) We remark that a general form for symbols of infinitesimal generators of Feller processes that arise as solutions of SDEs driven by Lévy processes in  $\mathbb{R}^n$  may be found in [40].

#### 4. Pseudo-Feller Jump Diffusions

We consider the linear operator  $\mathcal{A}$  defined on  $C^2(G)$  by

$$\mathcal{A}f(g) = b^i Y_i f(g) + \frac{1}{2} a^{ij} Y_i Y_j f(g) + \lambda \int_G (f(\tau) - f(g)) q(g, d\tau), \quad (5.19)$$

for  $f \in C^2(G)$ ,  $g \in G$  where the  $b_i$ s,  $a_{ij}$  and  $Y_i$ s are as in Example 3 and  $\lambda$  and  $q$  are as in Example 1. In fact we can write  $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$  where

$\mathcal{A}_1$  is as in (5.17) but with  $\nu \equiv 0$  and  $\mathcal{A}_2$  is as in (5.12). Since both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  generate one parameter contraction semigroups in  $C(G)$  so does  $\mathcal{A}$  (see Theorem 2.7 in [30], Chapter IX, p.501). Since both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  generate Feller semigroups, they satisfy the positive maximum principle and hence  $\mathcal{A}$  also satisfies this principle. It follows by the Hille-Yosida-Ray theorem that the closure of  $\mathcal{A}$  generates a Feller semigroup. We can easily compute the symbol of the generator to be:

$$\begin{aligned} j(g, \pi) &= b^i \gamma_i^j(g) d\pi(X_j) + \frac{1}{2} a^{ij} \gamma_i^k(g) [\gamma_j^r(g) d\pi(X_k) d\pi(X_r) \\ &+ X_k(g) \gamma_j^r(g) d\pi(X_r)] + \lambda \int_G (\pi(g^{-1}\tau) - I_\pi) q(g, d\tau). \end{aligned} \quad (5.20)$$

for each  $\pi \in \widehat{G}, g \in G$ .

##### 5. Subordination

Let  $(\rho_t^h, t \geq 0)$  be the law of a subordinator with associated Bernstein function  $h : (0, \infty) \rightarrow [0, \infty)$  so that  $(\rho_t^h, t \geq 0)$  is a vaguely continuous convolution semigroup of probability measures on  $[0, \infty)$  and for each  $t \geq 0, u > 0$ ,

$$\int_0^\infty e^{-us} \rho_t^h(ds) = e^{-th(u)}. \quad (5.21)$$

Note that  $h$  has the generic form

$$h(u) = bu + \int_{(0, \infty)} (1 - e^{-uy}) \lambda(dy),$$

where  $b \geq 0$  and  $\int_{(0, \infty)} (y \wedge 1) \lambda(dy) < \infty$  (see e.g. [38], section 30 and [6], section 1.3.2 for details.) Let  $(T_t, t \geq 0)$  be a Feller semigroup on  $C(G)$  for which  $\mathcal{M} \subseteq \text{Dom}(\mathcal{A})$ . By Lemma 4.1 and Theorem 4.2, for each  $t \geq 0$   $T_t$  and  $\mathcal{A}$  are pseudo differential operators and we denote their symbols by  $\sigma_t$  and  $j$  (respectively). Appealing to Phillip's theorem (see [35], [38] pp. 212-7) we see that subordination yields another Feller semigroup  $(T_t^h, t \geq 0)$  on  $C(G)$  with infinitesimal generator  $\mathcal{A}^h$  having  $\text{Dom}(\mathcal{A})$  as a core, where for each  $f \in C(G)$  we have (in the sense of Bochner integrals)

$$T_t^h f = \int_0^\infty T_s f \rho_t^h(ds),$$

and for each  $f \in \text{Dom}(\mathcal{A})$ ,

$$\mathcal{A}^h f = b\mathcal{A}f + \int_{(0, \infty)} (T_s f - f) \lambda^h(ds).$$

It then follows easily that  $\mathcal{A}^h$  is a pseudo differential operator and its symbol is given for  $g \in G, \pi \in \widehat{G}$  by

$$j^h(g, \pi) = bj(g, \pi) + \int_{(0, \infty)} (\sigma_s(g, \pi) - I_\pi) \lambda^h(ds).$$

## 6. Class $O$ Semigroups

We have already remarked that the Feller semigroups associated with convolution semigroups of measures commute with left translation on the group. In [4] a class of Feller semigroups was investigated under which there was an obstruction to these commutation relations. To be precise we require that there exists a measurable mapping  $h : \mathbb{R}^+ \times G \times G \times G \rightarrow \mathbb{R}$  for which

$$L_\rho T_t f(g) = T_t L_\rho f h(t, g, \rho, \cdot)(g),$$

for all  $t \geq 0, g, \rho \in G$ . Such semigroups were said to be of Class  $O$  (where the  $O$  stands for “obstruction”.) Define  $C'_2(G)$  in exactly the same way as  $C_2(G)$  but with the basis of left-invariant vector fields replaced by right-invariant ones. Under some technical conditions on  $h$  we have  $C'_2(G) \subseteq \text{Dom}(\mathcal{A})$  and (using the same notation where appropriate as was the case in Example 2) for  $f \in C'_2(G)$

$$\begin{aligned} \mathcal{A}f(g) &= b^i(g) X'_i f(g) + a^{ij}(g) X'_i X'_j f(g) \\ &+ \int_{G-\{e\}} [f(\tau) - f(g) - x^i(\tau g^{-1}) X'_i f(g)] \nu(g, d\tau) \end{aligned} \quad (5.22)$$

where  $b^i$  and  $a_{ij}$  are measurable functions on  $G$  ( $1 \leq i, j \leq n$ ) such that for each  $g \in G$ ,  $(a_{ij}(g))$  is a non-negative definite symmetric matrix and  $\nu(g, \cdot)$  is a Borel measure on  $G - \{e\}$  for which

$\int_{G-\{e\}} (\sum_{i=1}^n x_i(g^{-1}\tau)^2 \wedge 1) \nu(g, d\tau) < \infty$ . It follows that  $\mathcal{A}$  is a pseudo differential operator and by (4.11) we see that for all  $g \in G, \pi \in \widehat{G}$

$$\begin{aligned} j(g, \pi) &= b^i(g) \pi(g^{-1}) d\pi(X_i) \pi(g) + a^{ij}(g) \pi(g^{-1}) d\pi(X_i) d\pi(X_j) \pi(g) \\ &+ \int_{G-\{e\}} [\pi(g^{-1}\tau) - I_\pi - x^i(\tau g^{-1}) \pi(g^{-1}) d\pi(X_i) \pi(g)] \nu(g, d\tau) \end{aligned} \quad (5.23)$$

We remark that the form (5.22) was shown in [4] to hold for a class of Feller semigroups satisfying more general conditions (called “Hypothesis H” therein.)

## 6 Courrège-Hunt Operators and Dirichlet Forms

Motivated by the form of the symbols appearing in the examples, we seek a general class of Markov semigroups in  $L^2(G)$  whose infinitesimal generators are pseudo differential operators with symbol of the form

$$\begin{aligned} j(g, \pi) &= b^i(g)d\pi(X_i) + a^{ij}(g)d\pi(X_i)d\pi(X_j) \\ &+ \int_{G-\{e\}} [\pi(g^{-1}\tau) - I_\pi - x^i(g^{-1}\tau)d\pi(X_i)]\nu(g, d\tau), \end{aligned} \quad (6.24)$$

for  $g \in G, \pi \in \widehat{G}$ . We will specify the “characteristics”  $(b, a, \nu)$  more precisely below.

From now on we will take  $G$  to be an arbitrary  $n$ -dimensional Lie group equipped with a *right-invariant* Haar measure. We define a linear operator  $\mathcal{L}$  on  $L^2(G)$  by the prescription<sup>2</sup>

$$\begin{aligned} \mathcal{L}f(g) &= b^i(g)X_i f(g) + a^{ij}(g)X_i X_j f(g) \\ &+ \int_G [f(\tau) - f(g) - x^i(g^{-1}\tau)X_i f(g)]\nu(g, d\tau), \end{aligned} \quad (6.25)$$

for  $f \in \text{Dom}(\mathcal{L}), g \in G$ . We assume<sup>3</sup> that

- For each  $1 \leq i, j, k \leq n$ ,  $b^i$  and  $a^{jk}$  are bounded measurable functions on  $G$  and that for each  $g \in G$ ,  $(a^{jk}(g))$  is a non-negative definite symmetric matrix.
- For each  $1 \leq i \leq n$ ,  $x^i \in C_c^\infty(G)$  and  $(x_1, \dots, x_n)$  are a system of canonical co-ordinates for a given neighbourhood of the identity  $U$  in  $G$  that has compact closure.
- The mapping  $\nu : G \times \mathcal{B}(G) \rightarrow [0, \infty]$  is a *Lévy kernel* on  $G$  so that  $\nu(g, \{g\}) = 0$  for all  $g \in G$ , each  $\nu(g, \cdot)$  defines a Borel measure on  $G$ , the mapping  $g \rightarrow \nu(g, gE)$  is measurable for each  $E \in \mathcal{B}(G)$  and

$$\sup_{g \in G} \int_G \left( \sum_{i=1}^n x_i(\tau)^2 \wedge 1 \right) \nu(g, g d\tau) < \infty.$$

- There exists a Lévy measure  $\rho$  on  $G$  (with  $\rho(\{e\}) := 0$ ) such that  $\nu(g, g \cdot)$  is absolutely continuous with respect to  $\rho$  for all  $g \in G$ .

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<sup>2</sup>It is more convenient to work with real Hilbert spaces from now on.

<sup>3</sup>We make no claim that these conditions, or those that we will impose later to study  $\mathcal{L}^*$ , are in any sense optimal.

- $\sup_{\tau \in U^c} \sup_{g \in G} \lambda(g, \tau) < \infty$  where  $\lambda(g, \cdot)$  is the Radon-Nikodým derivative of  $\nu(g, g \cdot)$  with respect to  $\rho$ .
- There exists  $g' \in G$  such that  $\lambda(g', \tau) = \sup_{g \in G} \lambda(g, \tau)$  for all  $\tau \in U$ .

We call  $\mathcal{L}$  a *Courrège-Hunt operator* as its form is a natural extension to Lie groups of the operators on  $\mathbb{R}^n$  that are considered by Courrège in [11], while when the characteristics are constant we obtain the generators of convolution semigroups as described by Hunt [23]. Notice that the form of (6.25) is also very similar to that of (5.22) but we find it more convenient to work with left-invariant vector fields.

Let  $\mathcal{H}_2(G)$  be the Sobolev space (of order 2) defined by  $\mathcal{H}_2(G) := \{f \in L^2(G), X_i f \in L^2(G), X_j X_k f \in L^2(G), 1 \leq i, j, k \leq n\}$ . It is a real Banach space under the norm

$$|||f|||_2 = \left( \|f\|_2^2 + \sum_{i=1}^n \|X_i f\|_2^2 + \sum_{j,k=1}^n \|X_j X_k f\|_2^2 \right)^{\frac{1}{2}}.$$

Note that  $\mathcal{H}_2(G)$  is independent of the choice of basis used to define it. We will need the fact that  $C_c^\infty(G)$  is dense in  $\mathcal{H}_2(G)$ .

**Theorem 6.1**  $\mathcal{H}_2(G) \subseteq \text{Dom}(\mathcal{L})$ . In fact there exists  $C > 0$  such that for all  $f \in \mathcal{H}_2(G)$

$$\|\mathcal{L}f\|_2 \leq C |||f|||_2.$$

Hence  $\mathcal{L}$  is a bounded linear operator from  $\mathcal{H}_2(G)$  to  $L^2(G)$ .

*Proof.* Fix  $f \in C_c^\infty(G)$  and write  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$ , where  $\mathcal{L}_1 := b^i(\cdot)X_i$ ,  $\mathcal{L}_2 := a^{ij}(\cdot)X_i X_j$ ,  $\mathcal{L}_3 f(g) := \int_U [f(\tau) - f(g) - x^i(g^{-1}\tau)X_i f(g)]\nu(g, d\tau)$  for each  $g \in G$  and  $\mathcal{L}_4 := \mathcal{L} - \mathcal{L}_1 - \mathcal{L}_2 - \mathcal{L}_3$ . We will use the elementary inequality

$$\|\mathcal{L}\|_2^2 \leq 4(\|\mathcal{L}_1\|_2^2 + \|\mathcal{L}_2\|_2^2 + \|\mathcal{L}_3\|_2^2 + \|\mathcal{L}_4\|_2^2).$$

Straightforward use of the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|\mathcal{L}_1 f\|_2^2 &\leq \sum_{i=1}^n \int_G b^i(g)^2 \left( \sum_{j=1}^n X_j f(g)^2 \right) dg \\ &\leq n \max_{1 \leq i \leq n} \|b_i(g)\|_\infty^2 |||f|||_2^2, \end{aligned}$$

$$\begin{aligned} \|\mathcal{L}_2 f\|_2^2 &\leq \sum_{i,j=1}^n \int_G a_{ij}(g)^2 \left( \sum_{k,l=1}^n X_k X_l f(g)^2 \right) dg \\ &\leq n^2 \max_{1 \leq i,j \leq n} \|a_{i,j}(g)\|_\infty^2 |||f|||_2^2. \end{aligned}$$

We will find it convenient to make the change of variable  $\tau \rightarrow g\tau$  in the integrals defining both  $\mathcal{L}_3$  and  $\mathcal{L}_4$ . Then we have

$$\begin{aligned} \|L_4 f\|_2^2 &\leq 3 \int_G \left( \int_{U^c} f(g\tau) \nu(g, g d\tau) \right)^2 dg + 3 \int_G \left( \int_{U^c} f(g) \nu(g, g d\tau) \right)^2 dg \\ &\quad + 3 \int_G \left( \int_{U^c} \sum_{i=1}^n x^i(\tau) X_i f(g) \nu(g, g d\tau) \right)^2 dg. \end{aligned}$$

We consider each of the three integrals separately. For the second one it is easy to check that

$$\int_G \left( \int_{U^c} f(g) \nu(g, g d\tau) \right)^2 dg \leq \sup_{g \in G} \nu(g, g U^c)^2 \|f\|_2^2.$$

For the first integral, we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \int_G \left( \int_{U^c} f(g\tau) \nu(g, g d\tau) \right)^2 dg &\leq \sup_{g \in G} \nu(g, g U^c) \int_G \int_{U^c} f(g\tau)^2 \nu(g, g d\tau) dg \\ &= \sup_{g \in G} \nu(g, g U^c) \int_G \int_{U^c} f(g\tau)^2 \lambda(g, \tau) \rho(d\tau) dg \\ &\leq \sup_{g \in G} \nu(g, g U^c) \sup_{\tau \in U^c} \sup_{g \in G} \lambda(g, \tau) \rho(U^c) \|f\|_2^2, \end{aligned}$$

where we used Fubini's theorem and the right invariance of Haar measure to obtain the final estimate. Finally for the third integral we find that

$$\begin{aligned} &\int_G \left( \int_{U^c} \sum_{i=1}^n x^i(\tau) X_i f(g) \nu(g, g d\tau) \right)^2 dg \\ &\leq \sup_{\tau \in U^c} \left( \sum_{i=1}^n x_i(\tau)^2 \right) \sup_{g \in G} \nu(g, g U^c)^2 \int_G \left( \sum_{j=1}^n X_j f(g)^2 \right) dg \\ &\leq n \max_{1 \leq i \leq n} \|x_i\|_\infty^2 \sup_{g \in G} \nu(g, g U^c)^2 \|f\|_2^2. \end{aligned}$$

Combining the estimates for the three integrals together we can assert that there exists  $K > 0$  such that

$$\|\mathcal{L}_4 f\|_2 \leq K \|f\|_2.$$

We now turn our attention to  $\mathcal{L}_3$  and use a Taylor expansion as in [33] p.13 to write for each  $g \in G$ ,

$$\mathcal{L}_3 f(g) = \frac{1}{2} \int_U \sum_{i,j=1}^n x_i(\tau) x_j(\tau) X_i X_j f(g \epsilon_u(\tau)) \nu(g, g d\tau),$$

where  $0 < u < 1$  and  $\epsilon_u(\tau) := \exp(ux^i(\tau)X_i)$ . By repeated application of the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|\mathcal{L}_3 f\|_2^2 &\leq \frac{1}{4} \int_G \left[ \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \nu(g, g d\tau) \right) \times \right. \\
&\quad \left. \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \sum_{j,k=1}^n X_j X_k f(g\epsilon_u(\tau))^2 \nu(g, g d\tau) \right) \right] dg \\
&\leq \frac{1}{4} \sup_{g \in G} \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \nu(g, g d\tau) \right) \times \\
&\quad \left( \int_G \int_U \sum_{i=1}^n x_i(\tau)^2 \lambda(g', \tau) \sum_{j,k=1}^n X_j X_k f(g\epsilon_u(\tau))^2 \rho(d\tau) dg \right) \\
&\leq \frac{1}{4} \sup_{g \in G} \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \nu(g, g d\tau) \right) \times \\
&\quad \sup_{g' \in G} \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \nu(g', g' d\tau) \right) \int_G \sum_{j,k=1}^n X_j X_k f(g)^2 dg \\
&\leq \frac{1}{4} \sup_{g \in G} \left( \int_U \sum_{i=1}^n x_i(\tau)^2 \nu(g, g d\tau) \right)^2 \|f\|_2^2,
\end{aligned}$$

where we have again utilised Fubini's theorem and the right invariance of Haar measure.

Combining together the estimates for  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$  and  $\mathcal{L}_4$  we obtain the existence of  $C > 0$  such that  $\|\mathcal{L}f\|_2 \leq C\|f\|_2$  for all  $f \in C_c^\infty(G)$  and the required result follows by density.  $\square$

From now on we consider the Courrège-Hunt operator  $\mathcal{L}$  as a densely defined linear operator on  $L^2(G)$  with domain  $\mathcal{H}_2(G)$ .

**Corollary 6.1** *If  $G$  is compact then  $\mathcal{L}$  is a pseudo differential operator with symbol of the form (6.24).*

*Proof.* This is a consequence of the fact that  $\mathcal{M} \subseteq \mathcal{H}_2(G)$ .  $\square$

We would like to investigate the adjoint of  $\mathcal{L}$  and in order to do this we impose some additional assumptions:

- For all  $1 \leq i, j, k \leq n, b_i \in C_b^1(G)$  and  $a_{jk} \in C_b^2(G)$ .
- For all  $1 \leq i, j, k, l, p, q \leq n, X_l b_i \in B_b(G), X_l a_{jk} \in B_b(G)$  and  $X_p X_q a_{jk} \in B_b(G)$ .



- The mapping  $g \rightarrow \lambda(g, \tau) \in C^1(G)$  for  $\rho$ -almost all  $\tau \in G$ .

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$$\int_G \int_G |x^i(\tau) X_i \lambda(g, \tau)| \rho(d\tau) dg < \infty$$

and  $c \in B_b(G)$  where  $c(g) := \int_G x^i(\tau) X_i \lambda(g, \tau) \rho(d\tau)$  for all  $g \in G$ .

- $\lambda(g, \tau) > 0$  except on a possible set of  $\gamma_\nu$  measure zero where  $\gamma_\nu(d\tau, dg) := \nu(g, gd\tau) dg$ .
- There exists  $K > 0$  such that for all  $g \in G, \tau \in U$

$$|\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau)| \leq K |x^i(\tau) \alpha_i(g, \tau)|,$$

where  $\alpha_i \in B_b(G \times U)$  for  $1 \leq i \leq n$ .

We will assume that these assumptions hold for the remainder of this paper.

We define a linear operator  $R_\rho$  on  $L^2(G)$  by the prescription

$$R_\rho f(g) := \int_G f(g\tau^{-1}) (\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau)) \rho(d\tau).$$

for  $f \in \text{Dom}(R_\rho), g \in G$ . Straightforward manipulations yield

$$R_\rho f(g) = \int_G f(\tau^{-1}) \left( \frac{\lambda(\tau^{-1}, \tau g)}{\lambda(g, \tau g)} - 1 \right) \nu(g, gd\tau g).$$

The fact that  $C_c^\infty(G) \subseteq \text{Dom}(R_\rho)$  is demonstrated within the proof of the following theorem.

**Theorem 6.2** *We have  $C_c^\infty(G) \subseteq \text{Dom}(\mathcal{L}^*)$  and for all  $f \in C_c^\infty(G), g \in G$ ,*

$$\begin{aligned} \mathcal{L}^* f(g) &= c(g)f(g) - X^i(b_i f)(g) + X^i X^j(a_{ij} f)(g) \\ &= \int_G \left[ f(\tau^{-1}) \frac{\lambda(\tau^{-1}, \tau g)}{\lambda(g, \tau g)} - f(g) + x^i(\tau g) X_i f(g) \right] \nu(g, gd\tau g). \end{aligned} \tag{6.26}$$

*Proof.* Define a linear operator  $S$  on  $C_c^\infty(G)$  by the action on the right hand side of (6.26). We must show that  $\|Sf\|_2 < \infty$  for each  $f \in C_c^\infty(G)$ . We write  $S := S' + R_\rho^{(1)} + R_\rho^{(2)}$ , where for all  $g \in G$ ,

$$\begin{aligned} S' f(g) &:= c(g)f(g) - X^i(b_i f)(g) + X^i X^j(a_{ij} f)(g) \\ &= \int_G [f(\tau^{-1}) - f(g) + x^i(\tau g) X_i f(g)] \nu(g, gd\tau g), \end{aligned}$$

$R_\rho^{(1)}f(g) := \int_U f(g\tau^{-1})(\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau))\rho(d\tau)$  and  $R_\rho^{(2)} := R_\rho - R_\rho^{(1)}$ . By using similar arguments to those given in the proof of Theorem 6.1 we find that  $\|S'f\|_2 < \infty$  and  $\|R_\rho^{(2)}f\|_2 < \infty$ . Making repeated use of the Cauchy-Schwarz inequality we find that

$$\begin{aligned}
\|R_\rho^{(1)}f\|_2^2 &= \int_G \left( \int_U f(g\tau^{-1})(\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau))\rho(d\tau) \right)^2 dg \\
&\leq \int_G \left( \int_U f(g\tau^{-1})^2 |\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau)| \rho(d\tau) \right) \\
&\quad \times \left( \int_U |\lambda(g\tau^{-1}, \tau) - \lambda(g, \tau)| \rho(d\tau) \right) dg \\
&\leq K^2 \int_G \left( \int_U f(g\tau^{-1})^2 |x^i(\tau)\alpha_i(g, \tau)| \rho(d\tau) \right) \left( \int_U |x^i(\tau)\alpha_i(g, \tau)| \rho(d\tau) \right) \\
&\leq K^2 \sup_{g \in G, \tau \in U} \sum_{i=1}^n \alpha_i(g, \tau)^2 \left( \int_U \sum_{i=1}^n x^i(\tau)^2 \rho(d\tau) \right)^2 \|f\|_2^2 \\
&< \infty.
\end{aligned}$$

To show that  $S \subseteq \mathcal{L}^*$  its sufficient to consider the case  $b_i = a_{ij} = 0$  ( $1 \leq i, j \leq n$ ). Let  $(U_n, n \in \mathbb{N})$  be a sequence in  $\mathcal{B}(G)$  for which  $U_n \downarrow \{e\}$ .

We then find that for  $f, h \in C_c^\infty(G)$ , by use of the dominated convergence theorem

$$\begin{aligned}
&\langle \mathcal{L}h, f \rangle \\
&= \int_G \int_G (h(g\tau) - h(g) - x^i(\tau)X_i h(g)) f(g) \nu(g, g d\tau) dg \\
&= \int_G \int_G (h(g\tau) - h(g) - x^i(\tau)X_i h(g)) f(g) \lambda(g, \tau) \nu(d\tau) dg \\
&= \lim_{n \rightarrow \infty} \int_G \int_{G-U_n} (h(g\tau) - h(g) - x^i(\tau)X_i h(g)) f(g) \lambda(g, \tau) \nu(d\tau) dg.
\end{aligned}$$

The result follows from here by treating each term in the integrand separately and then passing to the limit. For example, making the change of variable  $g \rightarrow g\tau^{-1}$  we have for each  $n \in \mathbb{N}$ ,

$$\int_G \int_{G-U_n} h(g\tau) f(g) \nu(g, g d\tau) dg = \int_G \int_{G-U_n} h(g) f(g\tau^{-1}) \lambda(g\tau^{-1}, \tau) \rho(d\tau) dg.$$

Using the fact that for  $1 \leq i \leq n$ ,  $X_i$  acts as a derivation on  $C_c^\infty(G)$  we obtain

$$\mathcal{L}^* f(g) = c(g)f(g) + \int_G \left[ f(g\tau^{-1}) \frac{\lambda(g\tau^{-1}, \tau)}{\lambda(g, \tau)} - f(g) + x^i(\tau)X_i f(g) \right] \nu(g, g d\tau),$$

and the result follows when we make the change of variable  $\tau \rightarrow \tau g$  in the integral.  $\square$

We will find it convenient below to rewrite (6.26) in the form

$$\begin{aligned}\mathcal{L}^* f(g) &= c(g)f(g) + R_\rho f(g) - X^i(b_i f)(g) + X^i X^j(a_{ij} f)(g) \\ &= \int_G [f(\tau^{-1}) - f(g) + x^i(\tau g)X_i f(g)] \nu(g, g d\tau g),\end{aligned}\tag{6.27}$$

for each  $f \in C_c^\infty(G)$ ,  $g \in G$ . Since  $C_c^\infty(G) \subseteq \text{Dom}(\mathcal{L}) \cap \text{Dom}(\mathcal{L}^*)$ , we see that both  $\mathcal{L}$  and  $\mathcal{L}^*$  are closable in  $L^2(G)$ .

Note that  $\mathcal{L}^*$  is a pseudo-differential operator in the case when  $G$  is compact. Its symbol  $j'$  is easily calculated to be

$$\begin{aligned}j'(g, \pi) &= (c(g) - (X^i b_i)(g) + (X^i X^j a_{ij})(g))I_\pi \\ &\quad + (2(X^j a_{ij})(g) - b_i(g))d\pi(X^i) + a^{ij}(g)d\pi(X_i)d\pi(X_j) \\ &\quad + \int_G \left( \frac{\lambda(\tau^{-1}, \tau g)}{\lambda(g, \tau g)} \pi(g^{-1}\tau^{-1}) - I_\pi + x^i(\tau g)d\pi(X_i) \right) \nu(g, g d\tau g),\end{aligned}$$

for each  $g \in G, \pi \in \widehat{G}$ . For the remainder of this paper we will identify  $\mathcal{L}$  with its restriction to  $C_c^\infty(G)$ . We can and will choose  $(x_1, \dots, x_n)$  to be such that  $x_i(\tau^{-1}) = -x_i(\tau)$  for each  $\tau \in G, 1 \leq i \leq n$  (c.f. [7], p.219.)

**Theorem 6.3** *Suppose that the following conditions hold for all  $g \in G$ :*

1.  $b^i(g) = X_j(a^{ij})(g)$  for all  $1 \leq i \leq n$ .
2.  $R_\rho f(g) = -c(g)f(g)$  for all  $f \in C_c^\infty(G)$ .
3.  $\nu(g, gA) = \nu(g, gA^{-1})$  for all  $A \in \mathcal{B}(G)$ .

*Then  $\mathcal{L}$  is symmetric and we may write*

$$\begin{aligned}\mathcal{L}f(g) &= X^i(a_{ij}(g)X^j)f(g) \\ &\quad + \frac{1}{2} \int_G (f(g\tau) - 2f(g) + f(g\tau^{-1}))\nu(g, g d\tau),\end{aligned}\tag{6.28}$$

*for all  $f \in C_c^\infty(G), g \in G$ .*

*Proof.* Assume the hypotheses of the theorem hold. It is sufficient to consider the case  $a_{ij} = 0$  ( $1 \leq i, j \leq n$ ). Making the change of variable  $\tau \rightarrow g\tau^{-1}$  in (6.25) we obtain

$$\mathcal{L}f(g) = \int_G [f(g\tau^{-1}) - f(g) + x^i(\tau)X_i f(g)]\nu(g, g d\tau).$$

We recognise the last expression as  $\mathcal{L}^*f(g)$  when we make the change of variable  $\tau \rightarrow g\tau^{-1}$  in (6.27). Finally we obtain (6.28) by writing  $\mathcal{L}(f) = \frac{1}{2}(\mathcal{L}(f) + \mathcal{L}^*f)$ .  $\square$

When  $\mathcal{L}$  is symmetric and  $G$  is compact its symbol is given by

$$\begin{aligned} j(g, \pi) &= (X^i a_{ij})(g) d\pi(X^j) + a_{ij} d\pi(X^i) d\pi(X^j) \\ &+ \frac{1}{2} \int_G (\pi(\tau) - 2I_\pi + \pi(\tau^{-1})) \nu(g, g d\tau), \end{aligned}$$

for each  $g \in G, \pi \in \widehat{G}$ .

Note that when  $\nu \equiv 0$ , then condition (1) of Theorem 6.3 is both necessary and sufficient for  $\mathcal{L}$  to be symmetric and (6.28) is the well-known expression for a second order differential operator in divergence form. Suppose that condition (1) of Theorem 6.3 holds and that for each  $g \in G, A \in \mathcal{B}(G), \nu(g, A) = \rho(g^{-1}A)$  where  $\rho$  is (as above) a Lévy measure on  $G$ . In this case condition (3) of Theorem 6.3 is just the requirement that  $\rho$  is a symmetric measure and condition 2 holds automatically. We then have  $\mathcal{L} = \mathcal{L}_D + \mathcal{L}_J$  where  $\mathcal{L}_D$  is the infinitesimal generator of a diffusion process on  $G$  (in divergence form) and  $\mathcal{L}_J$  generates a pure jump Lévy process on  $G$ . It may be that this is the only possibility (however condition (2) in Theorem 6.3 is very strong) - indeed for future work on symmetric Courrège-Hunt operators it may be more fruitful to work in  $L^2(G, \mu)$  where  $\mu$  is an infinitesimal invariant measure for the linear operator  $\mathcal{L}$  (see [1]), but in that case the representation using pseudo differential operators will require further development.

We assume that  $\mathcal{L}$  is symmetric and define the symmetric bilinear form

$$\mathcal{E}(f_1, f_2) = -\langle \mathcal{L}f_1, f_2 \rangle,$$

for  $f_1, f_2 \in C_c^\infty(G)$ .

**Corollary 6.2** *The form  $\mathcal{E}$  has an extension which is a Dirichlet form on  $L^2(G)$ . It has the Beurling-Deny representation*

$$\begin{aligned} \mathcal{E}(f_1, f_2) &= \int_G a^{ij}(g) (X_i f_1)(g) (X_j f_2)(g) dg \\ &+ \frac{1}{2} \int_{(G \times G) - D} (f_1(\tau) - f_1(g)) (f_2(\tau) - f_2(g)) J(d\tau, dg), \end{aligned} \tag{6.29}$$

for  $f_1, f_2 \in C_c^\infty(G)$  where  $D := \{(g, g), g \in G\}$  and  $J(d\tau, dg) := \nu(g, d\tau) dg$ .

*Proof.* The representation (6.29) is obtained by standard manipulations (see e.g. Theorem 2.4 in [7] and Proposition 2.1 in [32] for the case where  $\mathcal{L}$  is the generator of a convolution semigroup.) It then follows that  $\mathcal{E}$  is Markovian by the argument given on page 7 of [15]. The form is closeable by standard arguments. Hence by Theorem 3.1.1 of [15], p.98 its smallest closed extension is a Dirichlet form.  $\square$

Some interesting relations between closability of  $\mathcal{E}$  and the behaviour of certain negligible sets can be deduced from results in [2].

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